On the Discrete Spectrum of Generalized Quantum Tubes

Christopher Lin and Zhiqin Lu*

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1 Introduction

The existence of discrete spectrum of the Laplacian on manifolds is an interesting phenomenon in both geometry and physics. In geometry, discrete spectrum characterizes compact manifolds as it is the single constituent for their spectrum, while it is often unknown that whether or not a complete noncompact manifold has discrete spectrum. On the other hand, in physics the discrete spectrum of the Laplacian represents quantization of energy of a free nonrelativistic particle and correspondingly the localization of the wave functions. The physics terminology of bound states are thus the eigenfunctions corresponding to points in the discrete spectrum.

The mathematical problem we study in this paper is the existence of the discrete spectrum of the Dirichlet Laplacian on a noncomplete noncompact manifold called the quantum tube, which is built as a type of tubular neighborhood about an immersed manifold in Euclidean space. The Dirichlet Laplacian is defined as the unique self-adjoint operator associated to the closed, positive symmetric quadratic form

$$Q(\psi, \varphi) = \int \langle \nabla\psi, \nabla\varphi \rangle \quad \forall \psi, \varphi \in C_{)}^{\infty}, \tag{1}$$

where we took the metric inner product of the gradients over the Riemannian manifold. Let Ω be a complete manifold with boundary. The Dirichlet Laplacian may be thought of as the Laplacian with Dirichlet boundary condition. In fact the space $\{f \mid f \in C^\infty(\Omega) \,\text{and}\, f|_{\partial\Omega} = 0\}$ is an operator core for the Dirichlet

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Laplacian. Let $\sigma(\Delta)$ and $\sigma_{ess}(\Delta)$ denote the total and essential spectrum of the Dirichlet Laplacian respectively. We have the following standard expressions for the bottom of the respective spectra on a noncompact manifold Ω:

$$\inf \sigma_{ess}(\Delta) = \sup_K \inf_{f \in C_0^\infty(\Omega \setminus K)} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}, \tag{2}$$

and

$$\inf \sigma(\Delta) = \inf_{f \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}, \tag{3}$$

where K are compact subsets of Ω.

As we mentioned, the discrete spectrum of the Laplacian may not exist for complete noncompact manifolds. However, in our previous paper [14] we showed that if we were to thicken a complete noncompact manifold by one more dimension, then under suitable curvature assumptions the discrete spectrum of the Dirichlet Laplacian on the resulting layer will be nonempty. In particular, we proved that $\inf \sigma(\Delta) < \inf \sigma_{ess}(\Delta)$ for the layer. In [14] the layer we studied is called the quantum layer, due to the fact that the additional dimension induces quantization of the spectrum. To be more precise, the quantum layer was defined as a tubular neighborhood about an orientable, complete noncompact manifold Σ of dimension $n \geq 2$, isometrically immersed in $\mathbb{R}^{n+1}$, with its width a suitably controlled by the second fundamental form of the immersion so that the tubular neighborhood is also an isometric immersion of $\Sigma \times (-a, a)$ into $\mathbb{R}^{n+1}$. In addition we required that Σ is parabolic (see section 2 for precise definition) and asymptotically flat. Then under suitable integral-curvature assumptions on Σ we can deduce that $\inf \sigma(\Delta) < \inf \sigma_{ess}(\Delta)$ for the quantum layer.[1]

In this paper, we will take the generalization further by allowing the codimension of the immersion of Σ to be arbitrary. Whereas in our last paper the geometric challenge in enlarging the dimension of the immersed manifold Σ was overcome by the use of parabolicity, the main geometric challenge presented here by the arbitrary codimension is to untangle the more complicated ways in which the geometry of Σ interacts with that of the ambient Euclidean space. In the quantum layer case, where the codimension is 1, the orientability of Σ allowed us to define an immersion $\mathcal{L} : \Sigma \times (-a, a) \longrightarrow \mathbb{R}^{n+1}$ by $\mathcal{L}(x, u) = x + uN$, where N is a global unit vector field on Σ. The existence of N on Σ creates a global vertical (or fiber) coordinate for the quantum layer. For codimension greater than 1, we no longer have such global fiber coordinates. Therefore to create a k-dimensional tubular neighborhood about Σ, we opt for a more intrinsic definition by lifting to the normal bundle (see definitions below). We will use the term quantum tube for this more generalized notion of a tubular

[1] The terminology of quantum layer was actually first introduced in the paper [4] by Duclos, Exner, and Krejčiřík. The inspiration comes from mesoscopic physics .

neighborhood about Σ. This new terminology brings to mind the image of a tube about a curve in space, although this paper does not deal with immersed one dimensional manifolds (for more remarks on this issue, see next section). On the quantum tube we will perform similar analysis as in [14] and prove again that the discrete spectrum is nonempty by showing $\inf \sigma(\Delta) < \inf \sigma_{ess}(\Delta)$, in particular the ground state (first eigenvalue) $\lambda_0 = \inf \sigma(\Delta)$ exists. However, we will see that the quantum tube has new geometric features that make this paper worthwile.

First, there is the problem of the nontriviality of the normal bundle of Σ if the codimension is greater than 1. For the quantum layer, where the normal bundle of Σ was trivial, to study the Laplacian on Σ we merely needed to concern ourselves with the tangent bundle of Σ and its connection. However, if the normal bundle is non-trivial, we may need to incorporate the normal bundle and the normal connection on Σ. In fact, the geometry of the quantum tube as revealed by its metric tensor, illustrates this point (see section 2). In the next section we will see that under our choice of a coordinate system, the metric tensor of the quantum tube will contain normal covariant derivatives of the fiber coordinate vectors along the horizontal coordinate frames. The metric tensor will be in non-block form as a result, which presents a difficulty in evaluating and estimating the volume measure on the quantum tube. One way to bypass this difficulty is to assume that these normal covariant derivatives vanish, which corresponds to the existence of parallel frames on Σ with respect to the normal connection. However, we know that on any vector bundle parallel frames exist if and only if the connection is flat, i.e., with identically zero curvature (see equation (1.1.4), Proposition 1.1.5, and Theorem 1.4.3 in [15]). To Assume that the immersion of Σ has flat normal bundle is a restrictive condition, even if the ambient manifold is Euclidean space. However, as we will see in sections 2 and 5, we do not need the assumption of flat normal bundle and the ensuing technical difficulty mentioned above could be resolved with some work.

The second feature of this paper is the further generalization of the curvature assumptions in [4] and [14]. Due to the arbitrary dimension and codimension of Σ, we had to consider the average of even elementary symmetric functions K_{2p} of the second fundamental form. These isometric invariants K_{2p} turn out to be quite well-known in the literature. We were informed by Mazzeo that these are the famous tube invariants of Weyl. A comprehensive book by Gray[8] discusses these tube invariants and their generalizations in much detail.

Next, we give a brief review of basic notions in isometric immersion. Let TN denote the tangent bundle of a manifold N, and T_pN its fiber at $p \in N$. Given an isometric immersion $\Sigma \hookrightarrow N$, we shall identify the image of the immersion with the manifold Σ itself. The Riemannian metric on N determines a sub-vector bundle of TN called the normal bundle of Σ which we denote by $T^{\perp}\Sigma$. At each point its fiber satisfies $T_p^{\perp}\Sigma \oplus T_p\Sigma = T_pN$. Formally we may define the normal bundle as the quotient bundle $T^{\perp}\Sigma = TN \diagup T\Sigma$. For the sake of convenience,

we will denote tangent vector fields on Σ by Roman letters and normal vector fields on Σ by Greek letters. Let ∇ be the Riemannnian connection on the ambient manifold N. Then the normal connection $\nabla^{\perp}$ on the normal bundle is defined by $\nabla^{\perp}_X \eta = (\nabla_X \eta)^{\perp}$. From the normal connection we also get the normal curvature tensor $R^{\perp}$ defined by $R^{\perp}(X,Y)\eta = \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + \nabla^{\perp}_{[X,Y]}\eta$.

We denote the second fundamental form of the immersion by $\vec{A}$, which is defined by the symmetric bilinear tensor $\vec{A}(X,Y) = (\nabla_X Y)^{\perp}$. The shape operator associated with a normal vector field η on N is the self-adjoint operator $S_\eta(X) = -(\nabla_X \eta)^T$, which is related to the second fundamental form by $\langle \vec{A}(X,Y), \eta \rangle = \langle S_\eta(X), Y \rangle$. Moreover, since the shape operator is a tensor we can define the elementary symmetric functions $C_j(S_\eta)$, $j = 1, 2, \cdots, n$, as genuine smooth functions on Σ and compute them locally.

Now, suppose the ambient manifold is the Euclidean space $\mathbb{R}^m$. Then it makes sense to define the map $f : T^{\perp}\Sigma \longrightarrow \mathbb{R}^m$ given by

$$f(x,\xi) = x + \xi. \tag{4}$$

The map (4) is well defined because every $\xi \in T^{\perp}_x\Sigma$ is naturally identified with a unique vector in $\mathbb{R}^m$ by $\xi = a_i \frac{\partial}{\partial x_i}$, where $\{x_1, \cdots, x_{n+k}\}$ are the standard coordinates of $\mathbb{R}^{n+k}$. f is also clearly smooth. Moreover, with the Euclidean metric ds^2_E we can compute the length $\|\xi\|$ of any normal vector ξ on Σ, and we can consider the bilinear norm $\|\vec{A}\|$ - which becomes a smooth function on Σ.

Let us introduce the following conditions on a complete manifold Σ:

A1) $\|\vec{A}\| \le \varepsilon_o$ for some $\varepsilon_o > 0$

A2) $\|\vec{A}\|(x) \to 0$ as $d(x, x_0) \to \infty$ for any fixed point $x_0 \in \Sigma$.

Definition 1. *Let $\Sigma \hookrightarrow \mathbb{R}^{n+k}$ be an isometric immersed, complete, noncompact n-dimensional oriented manifold with $n \ge 2$ satisfying condition A1). Let F be the submanifold of $T^{\perp}\Sigma$ defined by $F = \{(x,\xi) \in T^{\perp}\Sigma \mid \|\xi\| < r\}$, where r is any fixed positive number satifying $r \le (\sqrt{k}\,\varepsilon_o)^{-1}$. Then we define an order-k quantum tube with radius r as the Riemannian manifold $(F, f^*(ds^2_E))$, where $f^*(ds^2_E)$ is the pullback metric induced by the map f given in (1) restricted to F. We call Σ the base manifold of the quantum tube F.*

Our definition above requires that f restricted to F be an immersion into $\mathbb{R}^{n+k}$. The condition that guarantees this is exactly given by the requirement on the radius r, as we will see in section 2. Note that we do not require that F be embedded in Euclidean space, thus we allow self-intersections in $\mathbb{R}^{n+k}$, and the spectral results we derive still hold. It is also clear that our definition generalizes the definition given in [14] of the quantum layer. The definition we give above

will lend itself to features in the geometry of the quantum tube F that was not observed previously in the quantum layer of [14], as remarked earlier.

Having formally defined the quantum tube, we would like to pause to remark on how its peculiar structure allows us to study its spectrum. The quantum tube is clearly a noncompact, noncomplete manifold, and contrary to compact manifolds (with or without boundary) and complete manifolds, there are no standard techniques to study the Laplacian on such manifolds. However, the quantum tube is constructed so that there is the base manifold (the zero-section in the normal bundle) as the horizontal part and the fibers as the vertical part. In particular this allows us to construct test functions in the form of a product of a function on the base manifold and a function on the fibers. Therefore, we can more or less use special properties of the horizontal and vertical parts of the test function separately in the same analysis. As we will see in section 5, this allows us use special functions associated to the parabolicity on the base manifold (also see section 4) to construct a desired test function. Since the normal bundle may not be trivial, we only consider the vertical functions that are radially symmetric on each fiber.

We state the main results of this paper below.

Theorem 1. *Let $(F, f^*(ds_E^2))$ be an order-k quantum tube with radius r, with the base manifold Σ satisfying condition A2). Then the essential spectrum $\sigma_{ess}(\Delta)$ of the Dirichlet Laplacian Δ can be estimated from below as*

$$\inf \sigma_{ess}(\Delta) \geq \rho(k)^2/r^2, \tag{5}$$

where $\rho(k)^2 > 0$ is the first eigenvalue of the Dirichlet Laplacian on the k-dimensional Euclidean ball of radius $r = 1$.

Theorem 2. *Let $(F, f^*(ds_E^2))$ be an order-k quantum tube with radius r and base manifold Σ satisfying condition A1). We assume that Σ is parabolic and not totally geodesic. Suppose the function $\sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}$ is integrable and that*

$$\int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p} \;\leq\; 0, \tag{6}$$

where μ_{2p} are positive constant coefficients defined as

$$\mu_{2p} = p(2p+k-2) \int_0^1 t^{2p+k-3} \chi^2 \, dt;$$

$\chi = \chi(r)$ defines the eigenfunction of the first eigenvalue of the Dirichlet Laplacian on the k-dimensional Euclidean ball of radius 1:

$$\left\{ \begin{array}{l} \chi'' + \frac{n-1}{r}\chi = -\rho(k)^2 \chi \\ \chi'(0) = 0 \\ \chi(1) = 1 \end{array} \right. ; \tag{7}$$

K_{2p} are intrinsic curvature functions on Σ defined in Definition 4. Then the total spectrum $\sigma_0(\Delta)$ is strictly bounded above as

$$\inf \sigma_0(\Delta) \;<\; \rho(k)^2/r^2. \tag{8}$$

Combining the two theorems above we have the conclusive result:

Theorem 3. *Let $(F, f^*(ds_E^2))$ be an order-k quantum tube with radius r and base manifold Σ satisfying conditions A1) and A2). Moreover, we assume that Σ is a parabolic manifold with $\sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}$ integrable and*

$$\int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p} \;\leq\; 0,$$

If Σ is not totally geodesic, then the discrete spectrum of the Dirichlet Laplacian is non-empty.

Of particular interest is the surface case. Corollary 1.1 of [14] (or results in [1]) can be generalized to higher codimension without any rephrasing.

Corollary 1. *Suppose that Σ is a complete immersed surface of $\mathbf{R}^n (n \geq 2)$ such that the second fundamental form $\vec{A} \to 0$. Suppose that the Gauss curvature is integrable and suppose that*

$$e(\Sigma) - \sum \lambda_i \leq 0, \tag{9}$$

where λ_i is the isoperimetric constant at each end defined as follows. Let $E_1, \cdots, E_s$ be the ends of the surface Σ. For each E_i, we define

$$\lambda_i = \lim_{r \to \infty} \frac{A_i(r)}{\pi r^2}, \tag{10}$$

where $A_i(r)$ is the area of the ball $B(r) \cap E_i$. Let a be a positive number such that $a||A|| < C_0 < 1$. If Σ is not totally geodesic, then the ground state of the quantum layer Ω exists. In particular, if $e(\Sigma) \leq 0$, then the ground state exists.

Thus in the surface case the existence of discrete spectra can be controlled by the topology of the surface. This fact and its possible generalizations to higher dimensions should be interesting to pursue further.

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2 Geometry of Quantum Tubes

First we set up Fermi coordinate systems on quantum tubes to perform local computations. The coordinates are defined in the book [8].

Definition 2. *Consider the isometric immersion $\Sigma^n \hookrightarrow \mathbb{R}^{n+k}$ of the base manifold. For a local coordinate chart (U, φ) on Σ, we trivialize the normal bundle $T^\perp\Sigma$ over U with an orthonormal frame $\{\eta_1, \cdots, \eta_k\}$. Then for each $x \in U$, $(x, \xi) \in T^\perp_x\Sigma$ we define local coordinates $(x_1, \cdots, x_n, u_1, \cdots, u_k)$, where $\xi = u_\alpha \eta_\alpha(x)$. We call such a coordinate system a Femi coordinate system on $T^\perp\Sigma$.*

We now fix the convention of indexing the horizontal coordinates by Roman letters and fiber coordinates by Greek letters. Then for a Fermi coordinate system the entries in the metric tensor can be expressed as

$$f^*(ds_E^2) = G_{ij}dx_i dx_j + G_{i\alpha}dx_i du_\alpha + G_{\alpha i}du_\alpha dx_i + G_{\alpha\beta}du_\alpha du_\beta. \tag{11}$$

Then a straightforward but careful calculation gives

$$\begin{aligned} G_{ij} &= g_{ij} - 2u_\alpha\langle S_{\eta_\alpha}(\partial_i), \partial_j\rangle \\ &+ u_\alpha u_\beta\langle S_{\eta_\alpha}(\partial_i), S_{\eta_\beta}(\partial_j)\rangle + u_\alpha u_\beta\langle \nabla^\perp_{\partial_i}\eta_\alpha, \nabla^\perp_{\partial_j}\eta_\beta\rangle, \end{aligned} \tag{12}$$

$$G_{i\beta} = u_\alpha\langle \nabla^\perp_{\partial_i}\eta_\alpha, \eta_\beta\rangle, \tag{13}$$

$$G_{\alpha\beta} = \delta_{\alpha\beta}, \tag{14}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and $g_{ij}dx_i dx_j$ is the Riemann metric on Σ. Note that by our notations

$$u_\alpha u_\beta\langle \nabla^\perp_{\partial_i}\eta_\alpha, \nabla^\perp_{\partial_j}\eta_\beta\rangle = G_{i\gamma}G_{j\gamma}. \tag{15}$$

Compare to the metric tensor calculated in [14] for the quantum layer, here we see the appearance of terms involving the normal connection. Now, from the Frobenius theorem (see Theorem 1.4.3 in [15]) we know that the flatness of the normal bundle (i.e.,$R^\perp \equiv 0$ on Σ) guarantees that we can choose normal frames on Σ to annihilate these terms. The existence of these parallel normal frames (although not necessarily orthonormal) would simplify our analysis as it would reduce the metric tensor to block form similar to that of in [14]. However, as remarked in the introduction, this is too much to ask of immersions.

Remark 1. *Formulas* (12) *through* (15) *are also valid for the case where Σ is a curve ($n = 1$). It is interesting to note that in the case of a unit speed curve in $\mathbb{R}^3$, where we employ the usual Frenet frames, the terms $\nabla^\perp_{\partial_i}\eta_\alpha$ contain the torsion of the curve. One may wish to carry this intuition (and intuition only) further in the cases of our concern, namely $n \geq 2$. Then we will eventually see*

that the presence of torsion in the metric is in fact irrelevant for the existence of discrete spectra. Now, there is a series of pre-existing work on the spectral analysis of tubes built over curves (cf. [6, 7, 3, 2, 11]). A notable paper is [2], where the authors essentially demonstrated that the only criterion for the existence of discrete spectra is for the tube to be not straight but asymptotically straight. There they also proved that the essential spectrum is identical to that of the straight tube. It seems that the more intrinsic methods in this paper can also be used to analyze the curve case, but perhaps it is better to discuss this in a future paper.

From (12) we see that the $n \times n$ matrix (G_{ij}) can be written as

$$(G_{ij}) = (\tilde{G}_{ij}) + (\mathcal{N}_{ij}), \tag{16}$$

where

$$(\tilde{G}_{ij}) = (I - u_\alpha H_\alpha)^2 (g_{kj}), \tag{17}$$

and

$$(\mathcal{N}_{ij}) = (u_\alpha u_\beta \langle \nabla^\perp_{\partial_i} \eta_\alpha, \nabla^\perp_{\partial_j} \eta_\beta \rangle) \tag{18}$$

for the induced metric tensor (g_{ij}) on Σ and the local matrix H_α representing the shape operator ${S_{\eta_\alpha}}$[2].

Next, we let $(\tilde{G}_{ij})_{n\times n} = \tilde{G}$ and $(G_{i\beta})_{n\times k} = C$. Then in view of (14), (15), (16), and (17) we see that the metric tensor G on $T^\perp\Sigma$ with respect to the Fermi coordinate system has the block form

$$\begin{pmatrix} \tilde{G} + CC^T & C \\ C^T & I \end{pmatrix}. \tag{19}$$

The matrix C contains the normal connection terms, but we will see that the resulting volume element of the quantum tube will not depend on the normal connection explicitly.

Lemma 1. *Using the notations as above, we have* $\det G = \det \tilde{G}$

Proof. The trick is the following multiplication of matrices

$$\begin{pmatrix} \tilde{G} + CC^T & C \\ C^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -C^T & I \end{pmatrix} = \begin{pmatrix} \tilde{G} & C \\ 0 & I \end{pmatrix}. \tag{20}$$

□

Corollary 2. *The condition $r \leq (\sqrt{k}\,\varepsilon_o)^{-1}$ guarantees that the map f given by (1) is an immersion when restricted to the sub-bundle F, where $(F, f^*(ds^2_E))$ is an order-k quantum tube with radius r.*

[2] That is, $(H_\alpha)_{ij} = \langle S_{\eta_\alpha}(\partial_i), \partial_k \rangle g^{kj}$.

Proof. We know that the requirement that df be nonsingular at a point in $T^{\perp}\Sigma$ is equivalent to the condition that the pullback metric tensor G be a nonsingular matrix at the same point. By Lemma 1, this is reduced to knowing the invertibility of the matrix $\tilde{G}$. Suppose $\tilde{G}$ is invertible at a point, then

$$\tilde{G}^{-1} = (I - u_\alpha H_\alpha)^{-2} g^{-1} \tag{21}$$

from (17), where g denotes (g_{ij}). Note that $I - u_\alpha H_\alpha$ is invertible because g is. Now, we can write the Taylor expansion

$$(I - u_\alpha H_\alpha)^{-1} = \sum_{n=0}^{\infty} u_\alpha^n (H_\alpha)^n, \tag{22}$$

which converges if and only if

$$\| \sum_{\alpha=1}^{k} u_\alpha H_\alpha \| < 1. \tag{23}$$

On the other hand, we observe that

$$\| \sum_{\alpha=1}^{k} u_\alpha H_\alpha \| \quad \leq \quad \sum_{\alpha=1}^{k} |u_\alpha| \|H_\alpha\| \quad \leq \quad \|\vec{A}\| \sum_{\alpha=1}^{k} |u_\alpha| \quad \leq \quad \varepsilon_o \sum_{\alpha=1}^{k} |u_\alpha|$$

by condition A1) and the relation between the shape operator and the second fundamental form.

Therefore a sufficient condition for guaranteeing (23) is $\sum_{\alpha=1}^{k} |u_\alpha| < (\varepsilon_o)^{-1}$. However, to yield a geometrically invariant condition we further observe by the Cauchy inequality that

$$\sum_{\alpha=1}^{k} |u_\alpha| \quad \leq \quad \sqrt{k} \left(\sum_{\alpha=1}^{k} |u_\alpha|^2 \right)^{\frac{1}{2}}.$$

Thus we also obtain the sufficient condition $\left(\sum_{\alpha=1}^{k} |u_\alpha|^2 \right)^{\frac{1}{2}} < (\sqrt{k}\ \varepsilon_o)^{-1}$ that guarantees (23), and this is a statement about the distance to Σ, which is invariant under the choice of orthonormal frame in the normal bundle. Then by the definition of r, the condition $r \leq (\sqrt{k}\, \varepsilon_o)^{-1}$ ensures that the map f given by (4) is an immersion when restricted to the quantum tube.

□

Corollary 3. *Assume that $r < \frac{1}{\sqrt{k}}$. Then the volume element of an order-k quantum tube satisfies*

$$(1 - \varepsilon_o)^k \, du_A \, d\Sigma \quad \leq \quad \det(I - u_\alpha H_\alpha) \, du_A \, d\Sigma \quad \leq \quad (1 + \varepsilon_o)^k \, du_A \, d\Sigma, \tag{24}$$

where ε_o is the constant that appears in condition A1), and we assume $\varepsilon_0 < 1$.

Proof. By Lemma 1 and equation (17) we see that

$$
\begin{aligned}
dF &= \det(I - u_\alpha H_\alpha)\sqrt{g} du_A \, dx_J \\
&= \det(I - u_\alpha H_\alpha) du_A \, d\Sigma.
\end{aligned}
$$

The corollary follows from the fact that $\|u_\alpha H_\alpha\| < \varepsilon_0$. $\square$

We remark here that if the norm of the second fundamental form is bounded by any constant $0 < \varepsilon/\sqrt{k} < \varepsilon_0 < 1$, (24) still holds and we have

$$
(1-\varepsilon)^k \, du_A \, d\Sigma \quad \leq \quad \det(I - u_\alpha H_\alpha) \, du_A \, d\Sigma \quad \leq \quad (1+\varepsilon)^k \, du_A \, d\Sigma. \tag{25}
$$

The inequalities above will be key in the estimate of the lower bound of the essential spectrum.

We also need to calculate the inverse matrix G^{-1}. A straightforward computation using (20) gives

$$
\begin{aligned}
G^{-1} &= \begin{pmatrix} I & 0 \\ -C^T & I \end{pmatrix} \begin{pmatrix} \tilde{G}^{-1} & -\tilde{G}^{-1}C \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \tilde{G}^{-1} & -\tilde{G}^{-1}C \\ -C^T\tilde{G}^{-1} & C^T\tilde{G}^{-1}C + I \end{pmatrix} \\
&= \begin{pmatrix} \tilde{G}^{-1} & -\tilde{G}^{-1}C \\ -C^T\tilde{G}^{-1} & C^T\tilde{G}^{-1}C \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
\end{aligned} \tag{26}
$$

$\square$

We will end this section with one property of the above equation:

Lemma 2. $\begin{pmatrix} \tilde{G}^{-1} & -\tilde{G}^{-1}C \\ -C^T\tilde{G}^{-1} & C^T\tilde{G}^{-1}C \end{pmatrix}$ *is positive semidefinite.*

Proof. Let $\vec{v} = (v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{n+k}) = (\vec{v}_i \quad \vec{v}_\alpha) \in \mathbb{R}^{n+k}$, where $\vec{v}_i$ denotes the first n components and $\vec{v}_\alpha$ denotes the last k components. Then we see that

$$
\begin{aligned}
&(\vec{v}_i \quad \vec{v}_\alpha) \begin{pmatrix} \tilde{G}^{-1} & -\tilde{G}^{-1}C \\ -C^T\tilde{G}^{-1} & C^T\tilde{G}^{-1}C \end{pmatrix} \begin{pmatrix} \vec{v}_i \\ \vec{v}_\alpha \end{pmatrix} \\
&\quad = \tilde{G}^{ij} v_i v_j - 2\tilde{G}^{ij} G_{j\alpha} v_i v_\alpha + G_{\alpha i}\tilde{G}^{ij} G_{j\beta} v_\alpha v_\beta \\
&\quad = (\vec{v}_i - C\vec{v}_\alpha)^T \tilde{G}^{-1} (\vec{v}_i - C\vec{v}_\alpha).
\end{aligned} \tag{27}
$$

Since the matrix $\tilde{G}$ is clearly positive definite, the lemma is proved.

$\square$

3 Lower Bound Estimate of the Essential Spectrum

Let $f \in C_0^\infty(F)$. In the Fermi coordinate system, we have

$$|\nabla f|^2 \geq |\nabla^\perp f|^2 \tag{28}$$

by Lemma 2. The fiber-wise gradient $\nabla^\perp$ is essentially the gradient on the k-dimensional Euclidean ball.

It is clear that the boundary of a quantum tube F with radius r and base manifold Σ has as boundary the smooth manifold $\Sigma \times \partial B^k(0,r)$, where we identify the k-dimensional Euclidean open ball $B^k(0,r)$ with the fiber $F(x)$ at each $x \in \Sigma$. Then since $f = 0$ on ∂F, we have the Poincaré inequality

$$\int_{B^k(0,r)} |\nabla^\perp f|^2 \, du_A \quad \geq \quad \frac{\rho(k)^2}{r^2} \int_{B^k(0,r)} f^2 \, du_A, \tag{29}$$

where $\rho(k)^2$ is defined in Theorem 1.

Now we estimate the essential spectrum form below and prove Theorem 1.

Proof of Theorem 1. Fix a reference point x_o on Σ, condition A2) implies that for any $0 < \varepsilon < 1$ there exists an open ball $B(x_o, R) \subset \Sigma$ such that (25) holds outside its closure. We may then consider the compact set

$$K = \overline{B(x_o, R)} \times \overline{B^k(0,t)}$$

in F, where $t < r$. Using (25) and (28) we obtain

$$\frac{\int_F |\nabla f|^2}{\int_F f^2} \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^k \left(\frac{\int_F |\nabla^\perp f|^2 \;\; du_A \, d\Sigma}{\int_F f^2 \;\; du_A \, d\Sigma}\right) \tag{30}$$

for every $f \in C_0^\infty(F \setminus K)$. Then (29) implies that

$$\begin{aligned}
\int_F |\nabla^\perp f|^2 du_A \, d\Sigma &= \int_\Sigma \int_{B^k(0,r)} |\nabla^\perp f|^2 du_A \, d\Sigma \\
&\geq \;\; \frac{\rho(k)^2}{r^2} \int_\Sigma \int_{B^k(0,r)} f^2 \, du_A \, d\Sigma,
\end{aligned} \tag{31}$$

from which (30) becomes

$$\frac{\int_F |\nabla f|^2}{\int_F f^2} > \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^k \frac{\rho(k)^2}{r^2}. \tag{32}$$

for every $f \in C_0^\infty(F \setminus K)$. Then in view of (2) we obtain

$$\inf \sigma_{ess}(\Delta) \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^k \frac{\rho(k)^2}{r^2}. \tag{33}$$

Since ε is arbitrarily small, we obtain the desired estimate.

□

This result generalizes the lower bound estimate of the essential spectrum obtained in the codimension-1 case in [14], where $\rho(1)^2/r^2 = \kappa_1^2 = \frac{\pi^2}{4r^2}$.

4 Parabolicity of Complete Manifolds

In this section M will denote a complete manifold without boundary. The *parabolicity* on M was first defined as an analytic notion related to PDEs. Recall that a Green's function is the fundamental solution to Poisson's equation.

Definition 3. *M is parabolic if it does not admit any positive Green's function, otherwise it is said to be nonparabolic.*

The definition above and the subsequent material in this section is based on the survey paper by P. Li ([12]).

The definition given above for parabolicity may seem quite intangible. However, P. Li and L. F. Tam in [13] gave a procedure for constructing a Green's function on M. From the procedure, which involves applying the maximum principle to the positive Green's function on each Ω_i of a compact exhaustion $\{\Omega_i\}$ of M and analyzing the limit of such sequence of Green's functions, they could extract the following equivalent condition for parabolicity (see [12]):

Proposition 1. *Let $B(s)$ be a geodesic ball of radius s in M centered at any fixed point x_0. Let $R > s > 1$. Then let ψ_R be the solution to the following problem*

$$\begin{cases} \Delta\psi = 0 & \text{on } B(R) \setminus B(s); \\ \psi|_{B(s)} \equiv 1; \\ \psi|_{\Sigma \setminus B(R)} \equiv 0. \end{cases}$$

Then M is parabolic if and only if

$$\int_M |\nabla \psi_R|^2 \to 0 \quad \text{as } R \to \infty. \tag{34}$$

The harmonic functions $\{\psi_R\}$ is the key use of parabolicity in this paper, and will be used directly in the next section. The characterization of parabolicity is still anaytical in the proposition above. However, very often M being parabolic is also a geometric condition. First of all, there is the following result proven independently by Grigor'yan [9, 10] and Varopoulos [16]:

Theorem 4. *Let $V(t)$ be the volume of the geodesic ball $B(t)$ centered at any $p \in M$. If M is nonparabolic, then*

$$\int_1^{\infty} \frac{tdt}{V(t)} < \infty. \tag{35}$$

Therefore, if a geodesic ball is at most of quadratic growth then M must be parabolic.

In particular, the following corollary holds:

Corollary 4. *Any smooth surface with integrable Gaussian curvature must be parabolic.*

The corollary above is the reason why we did not have to assume the base manifold is parabolic in Corollary 1.

While the converse to Theorem 4 is not true by a counter-example of Greene (see [16], or more directly [12]), there is for example the following result.

Theorem 5 (Varopoulos [5]). *If M has non-negative Ricci curvature, then M is non-parabolic if and only if the volume growth condition (35) holds.*

Although its use is unnecessary, the theorem above implies that with the standard Euclidean metric, $\mathbb{R}^2$ is parabolic while $\mathbb{R}^n$ is nonparabolic for all $n \geq 3$. The reader can also view this distinction between $\mathbb{R}^2$ and $\mathbb{R}^n (n \geq 3)$ as a motivation for parabolicity (see our previous paper [14]). In gereral, if the volume growth of a complete manifold is quadratic, then it is parabolic.

5 Upper Bound Estimate of the Lower Bound of the Total Spectrum

We already saw in the previous two sections that the appearance of the normal connection terms $G_{i\alpha}$ did not stop us from generalizing the corresponding estimate for the codimension-1 case. The same will be true for the total spectrum.

First we recall from the introduction

$$\inf \sigma(\Delta) = \inf_{f\in C_0^\infty(F)} \frac{\int_F |\nabla f|^2}{\int_F f^2}. \tag{36}$$

By rescaling, we can assume the radius of F is 1. To prove Theorem 2, it then suffices to find a test function φ smooth almost everywhere on F such that

$$\int_F |\nabla\varphi|^2 - \rho(k)^2 \int_F \varphi^2 \quad < \quad 0. \tag{37}$$

As in [14], the prototype test function is of the form $\varphi = \chi\psi$, where χ depends only on the fiber and ψ depends only on the base manifold. Then since $\nabla\chi\psi = \chi\nabla\psi + \psi\nabla\chi$, we see that

$$|\nabla\varphi|^2 = |\nabla\chi\psi|^2 = \chi^2|\nabla\psi|^2 + \psi^2|\nabla\chi|^2 + 2\psi\chi\langle\nabla\psi, \nabla\chi\rangle. \tag{38}$$

Then (37) becomes

$$\int_F \psi^2(|\nabla\chi|^2 - \rho(k)^2\chi^2) + 2\int_F \psi\chi\langle\nabla\psi, \nabla\chi\rangle + \int_F \chi^2|\nabla\psi|^2 \quad < \quad 0. \tag{39}$$

We will assume that the base manifold is parabolic, so for a fixed point $x_o \in \Sigma$ and any $R > s > 1$, we can let the horizontal function be $\psi = \psi_R$ in Proposition 1, satisfying (34).

The choice of the fiber-wise function χ will also be similar to those in [14]. As we said in the introduction, χ should be radially symmetric in the fibers. Let t be the parameter for the length of each vector $\eta \in T_x^\perp\Sigma$. Since locally each fiber $F(x)$ is identified with $B^k(0,1)$, We will assume χ is of the form $\chi(t)$ on $B^k(0,1)$.

Then a straightforward computation using (26) shows that

$$|\nabla\chi|^2 = \Big(\frac{u_\alpha u_\beta}{t^2} G_{\alpha i}\tilde{G}^{ij}G_{j\beta} \ + \ 1\Big)\left|\frac{d\chi}{\partial t}\right|^2. \tag{40}$$

However, paying particular attention to the repeated indices for summation and using (13), we see that for each $j = 1, 2, \cdots, n$,

$$\begin{aligned} u_\beta G_{j\beta} &= u_\beta u_\gamma \langle \nabla^\perp_{\partial_j}\eta_\gamma, \eta_\beta\rangle \\ &= \sum_{\beta\neq\gamma} u_\beta u_\gamma \langle \nabla^\perp_{\partial_j}\eta_\gamma, \eta_\beta\rangle \\ &= 0, \end{aligned} \tag{41}$$

since $\{\eta_1, \cdots, \eta_k\}$ is an orthonormal frame and symmetry implies $\langle \nabla^{\perp}_{\partial_j} \eta_\gamma, \eta_\beta \rangle = -\langle \nabla^{\perp}_{\partial_j} \eta_\beta, \eta_\gamma \rangle$. Therefore (40) becomes

$$|\nabla \chi|^2 = \left| \frac{d\chi}{dt} \right|^2. \tag{42}$$

Using the same reasoning, we have

$$\langle \nabla \psi, \nabla \chi \rangle = 0 \tag{43}$$

if χ is rotationally symmetric.

Therefore, equation (41) again shows that the normal connection plays no explicit role in the analysis. Equation (39) now takes the familiar form as in [14]:

$$\int_F \psi^2 \left(\left| \frac{d\chi}{dt} \right|^2 - \rho(k)^2 \chi^2 \right) + \int_F \chi^2 |\nabla \psi|^2 \; < \; 0. \tag{44}$$

For the purpose of this section, we will be integrating in polar coordinates on the fibers. The use of polar coordinates is of course natural since our fibers are essentially Euclidean balls, although such a use was not necessary in proving Theorem 1. Now recall that for a point $(x, \xi) \in F$, we have

$$\det(I - u_\alpha H_\alpha) = \sum_{j=0}^{n} (-1)^j C_j(S_\xi). \tag{45}$$

In view of this we can write $\xi = t\eta$, where $\eta = \xi / \|\xi\|$. Then $S_\xi = tS_\eta$ and (45) becomes

$$\det(I - u_\alpha H_\alpha) \; = \; \det(I - tH_\eta) \; = \sum_{j=0}^{n} (-1)^j t^j C_j(S_\eta), \tag{46}$$

where H_η is the matrix of S_η with respect to the Fermi coordinate system and we define $C_0(S_\eta) = 1$.

The quantity $C_j(S_\eta)$ is a function on the unit sphere bundle

$$F_1 = \{(x, \eta) \in T^{\perp}\Sigma \mid \|\eta\| = 1\} \subset F.$$

Definition 4. *We define the following function on Σ (over all $\eta \in F_1(x)$):*

$$K_j = \int_{S^{k-1}} C_j(S_\eta) d\sigma,$$

and call it the jth-curvature of the quantum tube F, where S^{k-1} is identified with the fiber $F_1(x)$ and the integration is with respect to the boundary measure $d\sigma$ induced by the orientation of F.

An immediate observation from the definition is that all the odd-curvatures of F are 0. This is because we have $C_j(S_{-\eta}) = -C_j(S_\eta)$ and S^{k-1} is radially symmetric.

Let $\mathcal{R}$ be the curvature operator of Σ. We define the p-th trace of $\mathcal{R}$ as

$$\begin{aligned} tr(\mathcal{R}^p) &= \frac{1}{2^p((2p)!)^2} \\ \times \sum_I \sum_{\sigma,\tau\in S_{2p}} & sgn(\sigma\tau) R_{i_{\sigma(1)}i_{\sigma(2)}i_{\tau(1)}i_{\tau(2)}} \cdots R_{i_{\sigma(2p-1)}i_{\sigma(2p)}i_{\tau(2p-1)}i_{\tau(2p)}}. \end{aligned} \tag{47}$$

Then we have the following result from [8, equation (4.15)]:

Proposition 2. *For an isometric immersion of a manifold Σ with any codimension k into Euclidean space, we have at each point on Σ*

$$K_{2p} = \int_{S^{k-1}} C_{2p}(S_\eta) = \frac{(2p)!\pi^{k/2}}{2^{2p-1}p!\Gamma(p+k/2)} tr(\mathcal{R}^p). \tag{48}$$

□

The formula above is nice in that up to a multiple of a constant depending only on the dimesion and codimension, K_{2p} is determined entirely by the intrinsic Riemmanian structure of Σ.

Proof of Theorem 2. Using (12), condition A1), and (26), we can show that

$$\int_F \chi^2|\nabla\psi|^2 \,\le\, C_1 \int_\Sigma |\nabla_\Sigma\psi|^2 \tag{49}$$

for some constant $C_1 > 0$.

We consider the case

$$\int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}\, d\Sigma \,<\, 0. \tag{50}$$

first. Here we pick the prototype test function $\chi\psi$ described earlier.

Let ρ denote $\rho(k)$ and $B(0,1)$ denote the unit open ball in $\mathbb{R}^k$. Then using (46)

we have

$$
\begin{aligned}
&\int_F \psi^2 \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) det(I - u_\alpha H_\alpha)\, du_A\, d\Sigma \\
&\quad = \int_\Sigma \psi^2 \int_{B(0,1)} \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) \sum_{j=0}^{n} (-1)^j t^j C_j(S_\eta)\, du_A\, d\Sigma \\
&\quad = \int_\Sigma \psi^2 \sum_{j=1}^{n} (-1)^j \int_0^1 \int_{\partial B(0,t)} t^j \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) C_j(S_\eta)\, d\sigma_t\, dt\, d\Sigma \\
&\quad = \int_\Sigma \psi^2 \sum_{j=1}^{n} (-1)^j \int_0^1 t^j \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) dt \int_{\partial B(0,t)} C_j(S_\eta)\, d\sigma_t \;\, d\Sigma \\
&\quad = \int_\Sigma \psi^2 \sum_{j=1}^{n} (-1)^j \int_0^1 t^{j+k-1} \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) dt \int_{S^{k-1}} C_j(S_\eta)\, d\sigma \;\, d\Sigma \\
&\quad = \int_\Sigma \psi^2 \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}\, d\Sigma, \qquad (51)
\end{aligned}
$$

where for the fourth equality we used the fact that

$$
\int_{\partial B(0,t)} f\, d\sigma_t \;=\; t^{k-1} \int_{S^{k-1}} f\, d\sigma, \qquad (52)
$$

if f is independent of the radius parameter; and for the fifth equation we used (7).

By our choice of ψ, (50), and in view of (51), we can choose s large enough so that

$$
\int_F \psi^2 \left(\left| \frac{d\chi}{dt} \right|^2 - \rho^2 \chi^2 \right) \;<\; -\delta. \qquad (53)
$$

Moreover, by (49) and (34), we can then choose R large enough so that

$$
\int_F \chi^2 |\nabla \psi|^2 \;<\; \delta. \qquad (54)
$$

Combining (53) and (54) above yields (44), which proves the first part of the theorem.

Next we consider the case

$$
\int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}\, d\Sigma \;=\; 0. \qquad (55)
$$

For brevity we will define the quadratic form Q by

$$
Q(f,g) = \int_F \langle \nabla f, \nabla g \rangle \;-\; \rho^2 \int_F fg. \qquad (56)
$$

Then we consider as in [14] (and [4]) the same test function

$$\varphi_\varepsilon = \varphi + \varepsilon j \chi_1, \tag{57}$$

where ε is a small number, $\varphi = \chi\psi$ is the prototype test function, j a smooth function supported in $B(s) \subset \Sigma$, and χ_1 is some smooth function on $B(0,1)$ that vanishes on the boundary. In view of (36) and (56), it suffices to show that $Q(\varphi_\varepsilon, \varphi_\varepsilon) < 0$.

A quick computation shows that

$$Q(\varphi_\varepsilon, \varphi_\varepsilon) \;=\; Q(\varphi, \varphi) + 2\varepsilon Q(\varphi, j\chi_1) + \varepsilon^2 Q(j\chi_1, j\chi_1).$$

As in the previous case, we still have

$$Q(\varphi, \varphi) \;\leq\; C_1 \int_\Sigma |\nabla\psi|^2 \;+ C_2 \int_\Sigma \psi^2 \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p}. \tag{58}$$

Now, by the fact that $\mathrm{supp}(j) \subset B(s)$, and a careful calculation using the metric of F given by (19), we see that

$$Q(\varphi, j\chi_1) = \int_\Sigma j \int_{B(0,1)} \left(\frac{\partial\chi}{\partial u_\alpha} \frac{\partial\chi_1}{\partial u_\alpha} - \rho^2 \chi\chi_1 \right) \det(I - u_\alpha H_\alpha)\, du_\alpha\, d\Sigma. \tag{59}$$

Substituting the identity $\sum_{\alpha=1}^{k} \frac{\partial^2\chi}{\partial u_\alpha^2} = -\rho^2\chi$ on $B(0,1)$ into (59), and integrating by parts we see that

$$\begin{aligned} Q(\varphi, j\chi_1) &= -\int_\Sigma j \int_{B(0,1)} \chi_1 \frac{\partial\chi}{\partial u_\alpha} \frac{\partial}{\partial u_\alpha} \det(I - u_\alpha H_\alpha)\, du_\alpha\, d\Sigma \\ &= -\int_\Sigma j \int_{B(0,1)} \chi_1 \left\langle \nabla^\perp\chi, \nabla^\perp \det(I - u_\alpha H_\alpha) \right\rangle du_\alpha\, d\Sigma. \end{aligned} \tag{60}$$

Note that we used the fact that χ being radially symmetric implies it must have derivatives 0 at the origin.

From (46) we see that $\det(I - u_\alpha H_\alpha) = \det(I - tH_\eta)$, which is a polynomial in t at each $x \in \Sigma$. The assumption that Σ is not totally geodesic implies that there exist a point $x \in \Sigma$ on which there is an η such that $H_\eta \neq 0$. Then we can deduce that there must be a $t_0 \in (0, r)$ such that $\frac{\partial}{\partial t}\det(I - tH_\eta)(t_0) \neq 0$ at x. Moreover, since the zero set of χ is discrete, we can choose t_0 such that $\frac{d\chi}{dt}(t_0) \neq 0$ as well.

Now, using polar coordinates on $B(0,r)$ we have

$$\left\langle \nabla^\perp\chi, \nabla^\perp \det(I - u_\alpha H_\alpha) \right\rangle = \frac{d\chi}{dt} \frac{\partial}{\partial t} \det(I - tH_\eta),$$

which at $(x, t_0\eta)$ is not zero. Then we simply choose χ_1 to be a bump function about $t_0\eta$ and j a bump function about x, so that $Q(\varphi, j\chi_1) \neq 0$. Then we may choose a negative or positive ε small enough so that

$$2\varepsilon Q(\varphi, j\chi_1) + \varepsilon^2 Q(j\chi_1, j\chi_1) \; < \; -\delta. \tag{61}$$

With our choice of ψ and in view of (55) and (58), we see that we can choose s and R ($s < R$) such that

$$Q(\varphi, \varphi) < \delta. \tag{62}$$

Combining (61) and (62) we get $Q(\varphi_\varepsilon, \varphi_\varepsilon) < 0$, and the proof of the theorem is complete.

$\square$

We make a final remark on the following question brought up by the referee: other than tubes over totally geodesic submanifolds, are there quantum tubes that do not have pure point spectrum below the continuum?

The question is very difficult, even in the surface case. The following is a conjecture people are working on:

Conjecture. *Let Σ be a complete surface embedded into $\mathbf{R}^3$ such that the second fundamental form goes to zero at infinity. Suppose the Gaussian curvature is integrable. If there is no pure point spectrum below $\sigma_{ess}(\Delta)$ then Σ must be totally geodesic.*

Under the additional assumption that the Gaussian curvature is positive, the conjecture is true. This essentially follows from Theorem 1.3 of [14].

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